

Quiz 1

09/30/2026

Instructions:

- 1 Carefully go through each of the problems in the quiz. You have a total of 1 hour 15 minutes to solve this quiz.
- 2 You are allowed to assume basic laws of arithmetic including commutativity, associativity and all the axioms of a field for the Real numbers denoted by $\mathbb{R}$.
- 3 This quiz is worth 45 points.
- 4 Hints are given for more difficult problems, but you do not need to use them if you have another way to solve the problem.
- 5 Write your name clearly on each of the sheets. If you need extra sheets, please ask! If the extra sheets go to your final submission, please label which problem you are using it for and do not forget your name in the extra sheet.
- 6 Do not panic! We focus more on learning than scoring. It is absolutely fine if you are unable to solve all the problems completely.

Name: _____

JHED: _____

Problem 1 (5 points). *What is wrong with the following “proof”? We work in a field F that contains the integers $\mathbb{Z}$ and suppose $x, y \in F$. Let $x = y$. Then*

$$x^2 = xy,$$

$$x^2 - y^2 = xy - y^2,$$

$$(x + y)(x - y) = y(x - y),$$

$$x + y = y,$$

$$\text{since } x = y$$

$$2y = y,$$

$$2 = 1.$$

Problem 2 (10 points). *Suppose*

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \alpha.$$

In terms of α , find

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x}.$$

Prove your answer directly using the ϵ, δ definition. (Hint: What should be the limit $\lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$?)

Problem 3 (15 points). Let $\epsilon > 0$ be irrational. Prove that there exists a rational number q such that

$$0 < q < \epsilon.$$

Hint: You may use the fact that there exists a sequence of rational numbers $\{a_n\}$ converging to ϵ . Choose n sufficiently large that

$$|a_n - \epsilon| < \epsilon.$$

What does this tell you about a_n ? The number a_n may not be smaller than ϵ , but can you use the inequality above to get the desired rational number?

Problem 4 (15 points). (a) Prove that if $0 < a < 2$, then

$$a < \sqrt{2a} < 2.$$

(b) Prove that the sequence

$$\sqrt{2}, \quad \sqrt{2\sqrt{2}}, \quad \sqrt{2\sqrt{2\sqrt{2}}}, \quad \dots$$

converges.

(c) Find the limit. (Hint: Use the fact that if

$$\lim_{n \rightarrow \infty} a_n = l,$$

then

$$\lim_{n \rightarrow \infty} \sqrt{2a_n} = \sqrt{2l}.)$$
